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Research Article

A Functional Inequality in Restricted Domains of Banach Modules

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We investigate the stability problem for the following functional inequality $\|\alpha f((x+y)/2\alpha) + \beta f((y+z)/2\beta) + \gamma f((z+x)/2\gamma)\| \leq \|f(x+y+z)\|$ on restricted domains of Banach modules over a C^* -algebra. As an application we study the asymptotic behavior of a generalized additive mapping.

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1. Introduction and Preliminaries

The following question concerning the stability of group homomorphisms was posed by Ulam [1]: *Under what conditions does there exist a group homomorphism near an approximate group homomorphism?*

Hyers [2] considered the case of approximately additive mappings $f : E \rightarrow E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad (1.1)$$

for all $x, y \in E$.

In 1950, Aoki [3] provided a generalization of the Hyers' theorem for additive mappings and in 1978, Rassias [4] generalized the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded (see also [5]). The result of Rassias' theorem has been generalized by Forti [6, 7] and Gavruta [8] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers

have been published on the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9–23]). We also refer the readers to the books [24–28].

Throughout this paper, let A be a unital C^* -algebra with unitary group $U(A)$, unit e , and norm $|\cdot|$. Assume that \mathbb{X} is a left A -module and \mathbb{Y} is a left Banach A -module. An additive mapping $T : \mathbb{X} \rightarrow \mathbb{Y}$ is called A -linear if $T(ax) = aT(x)$ for all $a \in A$ and all $x \in \mathbb{X}$. In this paper, we investigate the stability problem for the following functional inequality:

$$\left\| \alpha f\left(\frac{x+y}{2\alpha}\right) + \beta f\left(\frac{y+z}{2\beta}\right) + \gamma f\left(\frac{z+x}{2\gamma}\right) \right\| \leq \|f(x+y+z)\| \quad (1.2)$$

on restricted domains of Banach modules over a C^* -algebra, where α, β, γ are nonzero positive real numbers. As an application we study the asymptotic behavior of a generalized additive mapping.

2. Solutions of the Functional Inequality (1.2)

Theorem 2.1. *Let \mathbb{X} and \mathbb{M} be left A -modules and let α, β, γ be nonzero real numbers. If a mapping $f : \mathbb{X} \rightarrow \mathbb{M}$ with $f(0) = 0$ satisfies the functional inequality*

$$\left\| \alpha f\left(\frac{ax+ay}{2\alpha}\right) + \beta f\left(\frac{ay+az}{2\beta}\right) + \gamma af\left(\frac{z+x}{2\gamma}\right) \right\| \leq \|f(ax+ay+az)\| \quad (2.1)$$

for all $x, y, z \in \mathbb{X}$ and all $a \in U(A)$, then f is A -linear.

Proof. Letting $z = -x - y$ in (2.1), we get

$$\alpha f\left(\frac{ax+ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma af\left(-\frac{y}{2\gamma}\right) = 0 \quad (2.2)$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Letting $x = 0$ (resp., $y = 0$) in (2.2), we get

$$\alpha f\left(\frac{ay}{2\alpha}\right) + \gamma af\left(-\frac{y}{2\gamma}\right) = 0, \quad \left(\text{resp., } \alpha f\left(\frac{ax}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) = 0\right) \quad (2.3)$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Hence $f(ay) = (-\gamma/\alpha)af((-a/\gamma)y)$ and it follows from (2.2) and (2.3) that $f((ax+ay)/2\alpha) - f(ax/2\alpha) - f(ay/2\alpha) = 0$ for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Therefore $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{X}$. Hence $f(rx) = rf(x)$ for all $x \in \mathbb{X}$ and all rational numbers r .

Now let $a \in A$ ($a \neq 0$) and let m be an integer number with $m > 4|a|$. Then by Theorem 1 of [29], there exist elements $u_1, u_2, u_3 \in U(A)$ such that $(3/m)a = u_1 + u_2 + u_3$. Since f is

additive and $f(rbx) = (-\gamma/\alpha)rbf((-\alpha/\gamma)x)$ for all $x \in \mathbb{X}$, all rational numbers r and all $b \in U(A)$, we have

$$\begin{aligned} f(ax) &= \frac{m}{3}f\left(\frac{3}{m}ax\right) = \frac{m}{3}f(u_1x + u_2x + u_3x) = \frac{m}{3}[f(u_1x) + f(u_2x) + f(u_3x)] \\ &= -\frac{m}{3}\frac{\gamma}{\alpha}(u_1 + u_2 + u_3)f\left(-\frac{\alpha}{\gamma}x\right) = -\frac{m}{3}\frac{\gamma}{\alpha}\frac{3}{m}af\left(-\frac{\alpha}{\gamma}x\right) = -\frac{\gamma}{\alpha}af\left(-\frac{\alpha}{\gamma}x\right) \end{aligned} \quad (2.4)$$

for all $x \in \mathbb{X}$. Replacing $(-\gamma/\alpha)x$ instead of x in the above equation, we have

$$f\left(-\frac{\gamma}{\alpha}ax\right) = -\frac{\gamma}{\alpha}af(x) \quad (2.5)$$

for all $x \in \mathbb{X}$. Since a is an arbitrary nonzero element in A in the previous paragraph, one can replace $(-\alpha/\gamma)a$ instead of a in (2.5). Thus we have $f(ax) = af(x)$ for all $x \in \mathbb{X}$ and all $a \in A$ ($a \neq 0$). So $f : \mathbb{X} \rightarrow \mathbb{Y}$ is A -linear. \square

The following theorem is another version of Theorem 2.1 on a restricted domain when $\alpha, \beta, \gamma > 0$.

Theorem 2.2. *Let \mathbb{X} and \mathbb{M} be left A -modules and let d, α, β, γ be nonzero positive real numbers. Assume that a mapping $f : \mathbb{X} \rightarrow \mathbb{M}$ satisfies $f(0) = 0$ and the functional inequality (2.1) for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq d$ and all $a \in U(A)$. Then f is A -linear.*

Proof. Letting $z = -x - y$ with $\|x\| + \|y\| \geq d$ in (2.1), we get

$$\alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma af\left(-\frac{y}{2\gamma}\right) = 0 \quad (2.6)$$

for all $a \in U(A)$. Let $\delta = \max\{|\beta|^{-1}d, |\gamma|^{-1}d\}$ and let $\|x\| + \|y\| \geq \delta$. Then

$$\|\beta x\| + \|\gamma y\| \geq \min\{|\beta|, |\gamma|\}(\|x\| + \|y\|) \geq \min\{|\beta|, |\gamma|\}\delta \geq d. \quad (2.7)$$

Therefore replacing x and y by $2\beta x$ and $2\gamma y$ in (2.6), respectively, we get

$$\alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) = 0 \quad (2.8)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq \delta$ and all $a \in U(A)$.

Similar to the proof of Theorem 3 of [30] (see also [31]), we prove that f satisfies (2.8) for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. Suppose $\|x\| + \|y\| < \delta$. If $\|x\| + \|y\| = 0$, let $z \in \mathbb{X}$ with $\|z\| = \delta$, otherwise

$$z := \begin{cases} (\delta + \|x\|) \frac{x}{\|x\|}, & \text{if } \|x\| \geq \|y\|; \\ (\delta + \|y\|) \frac{y}{\|y\|}, & \text{if } \|y\| \geq \|x\|. \end{cases} \quad (2.9)$$

Since $\alpha, \beta, \gamma > 0$, it is easy to verify that

$$\begin{aligned} & \left\| (2 + \beta^{-1}\gamma)z + \beta^{-1}\gamma y \right\| + \left\| \beta\gamma^{-1}x - (1 + 2\beta\gamma^{-1})z \right\| \geq \delta, \\ & \|x\| + \|z\| \geq \delta, \\ & \left\| 2(1 + \beta^{-1}\gamma)z \right\| + \|y\| \geq \delta, \\ & \left\| 2(1 + \beta^{-1}\gamma)z \right\| + \left\| \beta\gamma^{-1}x - (1 + 2\beta\gamma^{-1})z \right\| \geq \delta, \\ & \left\| (2 + \beta^{-1}\gamma)z + \beta^{-1}\gamma y \right\| + \|z\| \geq \delta. \end{aligned} \quad (2.10)$$

Therefore

$$\begin{aligned} & \alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) \\ &= \left[\alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f\left(-(2 + \beta^{-1}\gamma)az - \beta^{-1}\gamma ay\right) + \gamma af\left((1 + 2\beta\gamma^{-1})z - \beta\gamma^{-1}x\right) \right] \\ &+ \left[\alpha f\left(\frac{\beta ax + \gamma az}{\alpha}\right) + \beta f(-ax) + \gamma af(-z) \right] \\ &+ \left[\alpha f\left(\frac{2(\beta + \gamma)az + \gamma ay}{\alpha}\right) + \beta f(-2(1 + \beta^{-1}\gamma)az) + \gamma af(-y) \right] \\ &- \left[\alpha f\left(\frac{\beta ax + \gamma az}{\alpha}\right) + \beta f(-2(1 + \beta^{-1}\gamma)az) + \gamma af\left((1 + 2\beta\gamma^{-1})z - \beta\gamma^{-1}x\right) \right] \\ &- \left[\alpha f\left(\frac{2(\beta + \gamma)az + \gamma ay}{\alpha}\right) + \beta f(-(2 + \beta^{-1}\gamma)az - \beta^{-1}\gamma ay) + \gamma af(-z) \right] = 0. \end{aligned} \quad (2.11)$$

Hence f satisfies (2.8) and we infer that f satisfies (2.2) for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. By Theorem 2.1, f is A -linear. \square

3. Generalized Hyers-Ulam Stability of (1.2) on a Restricted Domain

In this section, we investigate the stability problem for A -linear mappings associated to the functional inequality (1.2) on a restricted domain. For convenience, we use the following abbreviation for a given function $f : \mathbb{X} \rightarrow \mathbb{Y}$ and $a \in U(A)$:

$$D_a f(x, y, z) := \alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(\frac{ay + az}{2\beta}\right) + \gamma af\left(\frac{z + x}{2\gamma}\right) \quad (3.1)$$

for all $x, y, z \in \mathbb{X}$.

Theorem 3.1. *Let $d, \alpha, \beta, \gamma > 0$, $p \in (0, 1)$, and $\theta, \varepsilon \geq 0$ be given. Assume that a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ satisfies the functional inequality*

$$f\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \theta + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad (3.2)$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq d$ and all $a \in U(A)$. Then there exist a unique A -linear mapping $T : \mathbb{X} \rightarrow \mathbb{Y}$ and a constant $C > 0$ such that

$$\|f(x) - T(x)\| \leq C + \frac{24 \times 2^p \alpha^{p-1} \varepsilon}{(2 - 2^p)} \|x\|^p \quad (3.3)$$

for all $x \in \mathbb{X}$.

Proof. Let $z = -x - y$ with $\|x\| + \|y\| \geq d$. Then (3.2) implies that

$$\begin{aligned} \left\| \alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma af\left(-\frac{y}{2\gamma}\right) \right\| &\leq \|f(0)\| + \theta + \varepsilon(\|x\|^p + \|y\|^p + \|x + y\|^p) \\ &\leq \|f(0)\| + \theta + 2\varepsilon(\|x\|^p + \|y\|^p). \end{aligned} \quad (3.4)$$

Thus

$$\left\| \alpha f\left(\frac{ax + ay}{\alpha}\right) + \beta f\left(-\frac{ax}{\beta}\right) + \gamma af\left(-\frac{y}{\gamma}\right) \right\| \leq \|f(0)\| + \theta + 2^{p+1} \varepsilon(\|x\|^p + \|y\|^p) \quad (3.5)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq d$ and all $a \in U(A)$. Let $\delta = \max\{\beta^{-1}d, \gamma^{-1}d\}$ and let $\|x\| + \|y\| \geq \delta$. Then $\|\beta x\| + \|\gamma y\| \geq d$. Therefore it follows from (3.5) that

$$\left\| \alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) \right\| \leq \|f(0)\| + \theta + 2^{p+1} \varepsilon(\|\beta x\|^p + \|\gamma y\|^p) \quad (3.6)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \geq \delta$ and all $a \in U(A)$. For the case $\|x\| + \|y\| < \delta$, let z be an element of \mathbb{X} which is defined in the proof of Theorem 2.2. It is clear that $\|z\| \leq 2\delta$. Using (2.11) and (3.6), we get

$$\begin{aligned}
& \left\| \alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) \right\| \\
& \leq \left\| \left[\alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f\left(-(2 + \beta^{-1}\gamma)az - \beta^{-1}\gamma ay\right) + \gamma af\left((1 + 2\beta\gamma^{-1})z - \beta\gamma^{-1}x\right) \right] \right\| \\
& \quad + \left\| \left[\alpha f\left(\frac{\beta ax + \gamma az}{\alpha}\right) + \beta f(-ax) + \gamma af(-z) \right] \right\| \\
& \quad + \left\| \left[\alpha f\left(\frac{2(\beta + \gamma)az + \gamma ay}{\alpha}\right) + \beta f\left(-2(1 + \beta^{-1}\gamma)az\right) + \gamma af(-y) \right] \right\| \\
& \quad + \left\| \left[\alpha f\left(\frac{\beta ax + \gamma az}{\alpha}\right) + \beta f\left(-2(1 + \beta^{-1}\gamma)az\right) + \gamma af\left((1 + 2\beta\gamma^{-1})z - \beta\gamma^{-1}x\right) \right] \right\| \\
& \quad + \left\| \left[\alpha f\left(\frac{2(\beta + \gamma)az + \gamma ay}{\alpha}\right) + \beta f\left(-(2 + \beta^{-1}\gamma)az - \beta^{-1}\gamma ay\right) + \gamma af(-z) \right] \right\| \\
& \leq 5(\|f(0)\| + \theta) + 4^{p+1}\varepsilon\delta^p [2(2\beta + \gamma)^p + 2^p(\beta + \gamma)^p + \gamma^p] + 6 \times 2^p\varepsilon(\|\beta x\|^p + \|\gamma y\|^p)
\end{aligned} \tag{3.7}$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| < \delta$ and all $a \in U(A)$. Hence

$$\left\| \alpha f\left(\frac{\beta ax + \gamma ay}{\alpha}\right) + \beta f(-ax) + \gamma af(-y) \right\| \leq K + 6 \times 2^p\varepsilon(\|\beta x\|^p + \|\gamma y\|^p) \tag{3.8}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$, where

$$K := 5(\|f(0)\| + \theta) + 4^{p+1}\varepsilon\delta^p [2(2\beta + \gamma)^p + 2^p(\beta + \gamma)^p + \gamma^p]. \tag{3.9}$$

Letting $x = 0$ and $y = 0$ in (3.8), respectively, we get

$$\begin{aligned}
& \left\| \alpha f\left(\frac{\gamma ay}{\alpha}\right) + \beta f(0) + \gamma af(-y) \right\| \leq K + 6 \times 2^p\varepsilon\|\gamma y\|^p, \\
& \left\| \alpha f\left(\frac{\beta ax}{\alpha}\right) + \beta f(-ax) + \gamma af(0) \right\| \leq K + 6 \times 2^p\varepsilon\|\beta x\|^p
\end{aligned} \tag{3.10}$$

for all $x, y \in \mathbb{X}$ and all $a \in U(A)$. It follows from (3.8) and (3.10) that

$$\|f(x + y) - f(x) - f(y)\| \leq \alpha^{-1}[(\beta + \gamma)\|f(0)\| + 3K + 12 \times 2^p\varepsilon(\|\alpha x\|^p + \|\alpha y\|^p)] \tag{3.11}$$

for all $x, y \in \mathbb{X}$. By the results of Hyers [2] and Rassias [4], there exists a unique additive mapping $T : \mathbb{X} \rightarrow \mathbb{Y}$ given by $T(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$ such that

$$\|f(x) - T(x)\| \leq \alpha^{-1}[(\beta + \gamma)\|f(0)\| + 3K] + \frac{24 \times 2^p \alpha^{p-1} \varepsilon}{(2 - 2^p)} \|x\|^p \quad (3.12)$$

for all $x \in \mathbb{X}$. It follows from the definition of T and (3.2) that $T(0) = 0$ and $\|D_a T(x, y, z)\| \leq \|T(ax + ay + az)\|$ for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq d$ and all $a \in U(A)$. Hence T is A -linear by Theorem 2.2. \square

We apply the result of Theorem 3.1 to study the asymptotic behavior of a generalized additive mapping. An asymptotic property of additive mappings has been proved by Skof [32] (see also [30, 33]).

Corollary 3.2. *Let α, β, γ be nonzero positive real numbers. Assume that a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ with $f(0) = 0$ satisfies*

$$\|D_a f(x, y, z) - f(ax + ay + az)\| \rightarrow 0 \quad \text{as } \|x\| + \|y\| + \|z\| \rightarrow \infty \quad (3.13)$$

for all $a \in U(A)$, then f is A -linear.

Proof. It follows from (3.13) that there exists a sequence $\{\delta_n\}$, monotonically decreasing to zero, such that

$$\|D_a f(x, y, z) - f(ax + ay + az)\| \leq \delta_n \quad (3.14)$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq n$ and all $a \in U(A)$. Therefore

$$\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \delta_n \quad (3.15)$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \geq n$ and all $a \in U(A)$. Applying (3.15) and Theorem 3.1, we obtain a sequence $\{T_n : \mathbb{X} \rightarrow \mathbb{Y}\}$ of unique A -linear mappings satisfying

$$\|f(x) - T_n(x)\| \leq 15\alpha^{-1}\delta_n \quad (3.16)$$

for all $x \in \mathbb{X}$. Since the sequence $\{\delta_n\}$ is monotonically decreasing, we conclude

$$\|f(x) - T_m(x)\| \leq 15\alpha^{-1}\delta_m \leq 15\alpha^{-1}\delta_n \quad (3.17)$$

for all $x \in \mathbb{X}$ and all $m \geq n$. The uniqueness of T_n implies $T_m = T_n$ for all $m \geq n$. Hence letting $n \rightarrow \infty$ in (3.16), we obtain that f is A -linear. \square

The following theorem is another version of Theorem 3.1 for the case $p > 1$.

Theorem 3.3. Let $p > 1, d > 0, \varepsilon \geq 0$ be given and let α, β, γ be nonzero real numbers. Assume that a mapping $f : \mathbb{X} \rightarrow \mathbb{Y}$ with $f(0) = 0$ satisfies the functional inequality

$$\|D_a f(x, y, z)\| \leq \|f(ax + ay + az)\| + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad (3.18)$$

for all $x, y, z \in \mathbb{X}$ with $\|x\| + \|y\| + \|z\| \leq d$ and all $a \in U(A)$. Then there exists a unique A -linear mapping $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ such that

$$\|\phi(x) - f(x)\| \leq \frac{(6 + 2^p) \times 2^p |\alpha|^{p-1} \varepsilon}{2^p - 2} \|x\|^p \quad (3.19)$$

for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$ and $\phi(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n}x)$.

Proof. Letting $z = -x - y$ in (3.18), we get

$$\left\| \alpha f\left(\frac{ax + ay}{2\alpha}\right) + \beta f\left(-\frac{ax}{2\beta}\right) + \gamma f\left(-\frac{y}{2\gamma}\right) \right\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|x + y\|^p) \quad (3.20)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \leq d/2$ and all $a \in U(A)$. Hence

$$\left\| \alpha f\left(\frac{ax + ay}{\alpha}\right) + \beta f\left(-\frac{ax}{\beta}\right) + \gamma f\left(-\frac{y}{\gamma}\right) \right\| \leq 2^p \varepsilon(\|x\|^p + \|y\|^p + \|x + y\|^p) \quad (3.21)$$

for all $x, y \in \mathbb{X}$ with $\|x\| + \|y\| \leq d/4$ and all $a \in U(A)$. It follows from (3.21) that

$$\begin{aligned} \left\| \alpha f\left(\frac{ax}{\alpha}\right) + \beta f\left(-\frac{ax}{\beta}\right) \right\| &\leq 2^{p+1} \varepsilon \|x\|^p, \\ \left\| \alpha f\left(\frac{ay}{\alpha}\right) + \gamma f\left(-\frac{y}{\gamma}\right) \right\| &\leq 2^{p+1} \varepsilon \|y\|^p \end{aligned} \quad (3.22)$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/4$ and all $a \in U(A)$. Adding (3.21) to (3.22), we get

$$\left\| \alpha f\left(\frac{ax + ay}{\alpha}\right) - \alpha f\left(\frac{ax}{\alpha}\right) - \alpha f\left(\frac{ay}{\alpha}\right) \right\| \leq 2^p \varepsilon (3\|x\|^p + 3\|y\|^p + \|x + y\|^p) \quad (3.23)$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/8$ and all $a \in U(A)$. Therefore

$$\|f(x + y) - f(x) - f(y)\| \leq 2^p |\alpha|^{p-1} \varepsilon (3\|x\|^p + 3\|y\|^p + \|x + y\|^p) \quad (3.24)$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\| \leq d/8|\alpha|$. Let $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$. We may put $y = x$ in (3.24) to obtain

$$\|f(2x) - 2f(x)\| \leq (6 + 2^p) \times 2^p |\alpha|^{p-1} \varepsilon \|x\|^p. \quad (3.25)$$

We can replace x by $x/2^{n+1}$ in (3.25) for all nonnegative integers n . Then using a similar argument given in [4], we have

$$\|2^{n+1}f(2^{-n-1}x) - 2^n f(2^{-n}x)\| \leq (6 + 2^p) \times \left(\frac{2}{2^p}\right)^n |\alpha|^{p-1} \varepsilon \|x\|^p. \quad (3.26)$$

Hence we have the following inequality:

$$\begin{aligned} \|2^{n+1}f(2^{-n-1}x) - 2^m f(2^{-m}x)\| &\leq \sum_{k=m}^n \|2^{k+1}f(2^{-k-1}x) - 2^k f(2^{-k}x)\| \\ &\leq (6 + 2^p) |\alpha|^{p-1} \varepsilon \sum_{k=m}^n \left(\frac{2}{2^p}\right)^k \|x\|^p \end{aligned} \quad (3.27)$$

for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$ and all integers $n \geq m \geq 0$. Since Y is complete, (3.27) shows that the limit $T(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n}x)$ exists for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$. Letting $m = 0$ and $n \rightarrow \infty$ in (3.27), we obtain that T satisfies inequality (3.19) for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$. It follows from the definition of T and (3.24) that

$$T(x + y) = T(x) + T(y) \quad (3.28)$$

for all $x, y \in \mathbb{X}$ with $\|x\|, \|y\|, \|x + y\| \leq d/8|\alpha|$. Hence

$$T\left(\frac{x}{2}\right) = \frac{1}{2}T(x) \quad (3.29)$$

for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$. We extend the additivity of T to the whole space \mathbb{X} by using an extension method of Skof [34]. Let $\delta := d/8|\alpha|$ and $x \in \mathbb{X}$ be given with $\|x\| > \delta$. Let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \leq 2^k\delta$. We define the mapping $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ by

$$\phi(x) := \begin{cases} T(x), & \text{if } \|x\| \leq \delta, \\ 2^k T(2^{-k}x), & \text{if } \|x\| > \delta. \end{cases} \quad (3.30)$$

Let $x \in \mathbb{X}$ be given with $\|x\| > \delta$ and let $k = k(x)$ be the smallest integer such that $2^{k-1}\delta < \|x\| \leq 2^k\delta$. Then $k-1$ is the smallest integer satisfying $2^{k-2}\delta < \|x/2\| \leq 2^{k-1}\delta$. If $k = 1$, we have $\phi(x/2) = T(x/2)$ and $\phi(x) = 2T(x/2)$. Therefore $\phi(x/2) = (1/2)\phi(x)$. For the case $k > 1$, it follows from the definition of ϕ that

$$\phi\left(\frac{x}{2}\right) = 2^{k-1}T\left(2^{-(k-1)}\frac{x}{2}\right) = \frac{1}{2} \cdot 2^k T(2^{-k}x) = \frac{1}{2}\phi(x). \quad (3.31)$$

From the definition of ϕ and (3.29), we get that $\phi(x/2) = (1/2)\phi(x)$ holds true for all $x \in \mathbb{X}$. Let $x \in \mathbb{X}$ and let k be an integer such that $\|x\| \leq 2^k\delta$. Then

$$\phi(x) = 2^k \phi(2^{-k}x) = 2^k T(2^{-k}x) = \lim_{n \rightarrow \infty} 2^{n+k} f(2^{-(n+k)}x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n}x). \quad (3.32)$$

It remains to prove that ϕ is A -linear. Let $x, y \in \mathbb{X}$ and let n be a positive integer such that $\|x\|, \|y\|, \|x+y\| \leq 2^n\delta$. Since $\phi(x/2) = (1/2)\phi(x)$ for all $x \in \mathbb{X}$ and T satisfies (3.28), we have

$$\begin{aligned} \phi(x+y) &= 2^n \phi\left(\frac{x+y}{2^n}\right) = 2^n T\left(\frac{x+y}{2^n}\right) = 2^n \left[T\left(\frac{x}{2^n}\right) + T\left(\frac{y}{2^n}\right)\right] \\ &= 2^n \left[\phi\left(\frac{x}{2^n}\right) + \phi\left(\frac{y}{2^n}\right)\right] = \phi(x) + \phi(y). \end{aligned} \quad (3.33)$$

Hence ϕ is additive. Since $\phi(x) = \lim_{n \rightarrow \infty} 2^n f(2^{-n}x)$ for all $x \in \mathbb{X}$, we have from (3.22) that $\alpha\phi(ay/\alpha) = \gamma a\phi(y/\gamma)$ for all $y \in \mathbb{X}$ and all $a \in U(A)$. Letting $a = e$, we get $\alpha\phi(y/\alpha) = \gamma\phi(y/\gamma)$. Therefore $\phi(ay) = a\phi(y)$ for all $y \in \mathbb{X}$ and all $a \in U(A)$. This proves that ϕ is A -linear. Also, ϕ satisfies inequality (3.19) for all $x \in \mathbb{X}$ with $\|x\| \leq d/8|\alpha|$, by the definition of ϕ . \square

For the case $p = 1$ we use the Gajda's example [35] to give the following counterexample.

Example 3.4. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} x, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases} \quad (3.34)$$

Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x). \quad (3.35)$$

It is clear that f is continuous, bounded by 2 on \mathbb{C} and

$$|f(x+y) - f(x) - f(y)| \leq 6(|x| + |y|) \quad (3.36)$$

for all $x, y \in \mathbb{C}$ (see [35]). It follows from (3.36) that the following inequality:

$$|f(x+y+z) - f(x) - f(y) - f(z)| \leq 12(|x| + |y| + |z|) \quad (3.37)$$

holds for all $x, y, z \in \mathbb{C}$. First we show that

$$|f(\lambda x) - \lambda f(x)| \leq 2(1 + |\lambda|)^2 |x| \quad (3.38)$$

for all $x, \lambda \in \mathbb{C}$. If f satisfies (3.38) for all $|\lambda| \geq 1$, then f satisfies (3.38) for all $\lambda \in \mathbb{C}$. To see this, let $0 < |\lambda| < 1$ (the result is obvious when $\lambda = 0$). Then $|f(\lambda^{-1}x) - \lambda^{-1}f(x)| \leq 2(1 + |\lambda|^{-1})^2|x|$ for all $x \in \mathbb{C}$. Replacing x by λx , we get that $|f(\lambda x) - \lambda f(x)| \leq 2|\lambda|^2(1 + |\lambda|^{-1})^2|x| = 2(1 + |\lambda|)^2|x|$ for all $x \in \mathbb{C}$. Hence we may assume that $|\lambda| \geq 1$. If $\lambda x = 0$ or $|\lambda x| \geq 1$, then

$$|f(\lambda x) - \lambda f(x)| \leq 2(1 + |\lambda|) \leq 2|\lambda|(1 + |\lambda|)|x| \leq 2(1 + |\lambda|)^2|x|. \quad (3.39)$$

Now suppose that $0 < |\lambda x| < 1$. Then there exists an integer $k \geq 0$ such that

$$\frac{1}{2^{k+1}} \leq |\lambda x| < \frac{1}{2^k}. \quad (3.40)$$

Therefore

$$2^k|x|, 2^k|\lambda x| \in (-1, 1). \quad (3.41)$$

Hence

$$2^m|x|, 2^m|\lambda x| \in (-1, 1) \quad (3.42)$$

for all $m = 0, 1, \dots, k$. From the definition of f and (3.40), we have

$$\begin{aligned} |f(\lambda x) - \lambda f(x)| &= \left| \sum_{n=k+1}^{\infty} \frac{1}{2^n} [\phi(2^n \lambda x) - \lambda \phi(2^n x)] \right| \\ &\leq (1 + |\lambda|) \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1 + |\lambda|}{2^k} \leq 2|\lambda|(1 + |\lambda|)|x| \leq 2(1 + |\lambda|)^2|x|. \end{aligned} \quad (3.43)$$

Therefore f satisfies (3.38). Now we prove that

$$\begin{aligned} &|D_{\mu}f(x, y, z) - f(\mu x + \mu y + \mu z)| \\ &\leq \left(16 + |\alpha|^{-1}(1 + |\alpha|)^2 + |\beta|^{-1}(1 + |\beta|)^2 + |\gamma|^{-1}(1 + |\gamma|)^2\right) (|x| + |y| + |z|) \end{aligned} \quad (3.44)$$

for all $x, y, z \in \mathbb{C}$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, where

$$D_{\mu}f(x, y, z) := \alpha f\left(\frac{\mu x + \mu y}{2\alpha}\right) + \beta f\left(\frac{\mu y + \mu z}{2\beta}\right) + \gamma \mu f\left(\frac{z + x}{2\gamma}\right). \quad (3.45)$$

It follows from (3.37) and (3.38) that

$$\begin{aligned}
 & |D_\mu f(x, y, z) - f(\mu x + \mu y + \mu z)| \\
 & \leq \left| \alpha f\left(\frac{\mu x + \mu y}{2\alpha}\right) - f\left(\frac{\mu x + \mu y}{2}\right) \right| + \left| \beta f\left(\frac{\mu y + \mu z}{2\beta}\right) - f\left(\frac{\mu y + \mu z}{2}\right) \right| \\
 & \quad + \left| \gamma \mu f\left(\frac{z+x}{2\gamma}\right) - \mu f\left(\frac{z+x}{2}\right) \right| + \left| \mu f\left(\frac{z+x}{2}\right) - f\left(\frac{\mu z + \mu x}{2}\right) \right| \\
 & \quad + \left| f\left(\frac{\mu x + \mu y}{2}\right) + f\left(\frac{\mu y + \mu z}{2}\right) + f\left(\frac{\mu z + \mu x}{2}\right) - f(\mu x + \mu y + \mu z) \right| \\
 & \leq (6 + |\alpha|^{-1}(1 + |\alpha|)^2)|x + y| + (6 + |\beta|^{-1}(1 + |\beta|)^2)|y + z| + (10 + |\gamma|^{-1}(1 + |\gamma|)^2)|x + z| \\
 & \leq (16 + |\alpha|^{-1}(1 + |\alpha|)^2 + |\beta|^{-1}(1 + |\beta|)^2 + |\gamma|^{-1}(1 + |\gamma|)^2)(|x| + |y| + |z|)
 \end{aligned} \tag{3.46}$$

for all $x, y, z \in \mathbb{C}$ and all $\mu \in \mathbb{T}^1$. Thus f satisfies inequality (3.18) for $p = 1$. Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a linear functional such that

$$|f(x) - T(x)| \leq M|x| \tag{3.47}$$

for all $x \in \mathbb{C}$, where M is a positive constant. Then there exists a constant $c \in \mathbb{C}$ such that $T(x) = cx$ for all rational numbers x . So we have

$$|f(x)| \leq (M + |c|)|x| \tag{3.48}$$

for all rational numbers x . Let $m \in \mathbb{N}$ with $m > M + |c|$. If $x_0 \in (0, 2^{-m+1}) \cap \mathbb{Q}$, then $2^n x_0 \in (0, 1)$ for all $n = 0, 1, \dots, m-1$. So

$$f(x_0) \geq \sum_{n=0}^{m-1} \frac{1}{2^n} \phi(2^n x_0) = m x_0 > (M + |c|)x_0, \tag{3.49}$$

which contradicts (3.48).

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